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Bubble Coalescence in Fluidized Beds

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The two-dimensional motion of two circular bubbles moving in train up through a fluidized bed is considered. By using continuum equations of motion for this two-phase flow situation, neutral coalescence curves are obtained which indicate, for given bubble radii and distance apart, whether or not the bubbles will coalesce. The conclusions are in agreement with the available experimental results.

In most gas-particle fluidized beds, bubbles or voids of particles pass up through the bed. Empirically it appears that bubbles occur when the ratio of the solid to gas densities is greater than about 10. The gas and particulate motion resulting from the motion of these bubbles is of major importance in particle mixing and in understanding the nature of particle-gas contact. Considerable work has been done on the motion of single bubbles in fluidized beds. Of particular interest from the more fundamental experimental side is the work by Rowe and Partridge and their associates at the Atomic Energy Research Establishment at Harwell. Papers by Rowe and Partridge (*12*) on x-ray studies of bubbles and by Rowe, Partridge, and Lyall (*13*) also give references to previous work and other relevant papers on the experimental side. The latter paper also gives a careful comparison between theory and experiment as of that date. Harrison and Leung (*4*) [see also Davidson and Harrison (*2*)] carried out preliminary experimental work specifically on the coalescence of two-dimensional bubbles moving in train up through a fluidized bed. Their experimental results are compared with the results derived in this paper.

Theoretical studies on the motion (and resulting flows) of single bubbles have been given by Davidson (*1*) [see also Davidson and Harrison (*2*) who give a thorough review of fluidization up to 1962 from both the experimental and theoretical side], Jackson (*5*), and Murray (*7* to *10*). All of these consider continuum flows for both the fluid and particles. Recently, Murray (*10*) considered the unsteady bubble motion investigated experimentally by Partridge and Lyall (*11*); fairly good agreement was found. The flow fields in these situations were shown to be potential flows with constant voidage throughout. The approximate equations he deduced for such situations are linear (because of the Oseenlike approach), simple, and very easy to use. These are given below for the situation under consideration here.

The interesting work by Toei and Matsuno (*14*), which came out after completion of the work in this paper, also treats coalescence theoretically as well as experimentally. Unfortunately, their steady state solutions, obtained on setting $\partial v_s/\partial t = 0$, are not steady since even when the bubbles are in a vertical line, the velocities they find for them are different and thus their distance apart is a function of time, which is an unsteady situation.

EQUATIONS OF MOTION AND SOLUTION

Murray (*9*, *7*) suggested approximate equations of motion for the steady state continuum flow of solids and gas phases in most fluidized beds. These are, in dimensionless form

$$\mathrm{div}\,(1-\epsilon)\mathbf{v}_s = 0 = \mathrm{div}\,\epsilon\mathbf{v}_f,$$

$$(1-\epsilon)(\mathbf{v}_s\cdot\mathrm{grad})\mathbf{v}_s = -(1-\epsilon)\mathbf{i}/F + D(\epsilon)(\mathbf{v}_f-\mathbf{v}_s)$$

$$\mathrm{grad}\ p = -D(\epsilon)(\mathbf{v}_f-\mathbf{v}_s) \tag{1}$$

The first two equations are mass, while the second two are momentum, conservation equations.

With the modified Oseen linearization and the resulting constant voidage solution, reduce to the following linear equations [see Murray (9)]:

$$\mathrm{div}\ \mathbf{v}_s = 0 = \mathrm{div}\ \mathbf{v}_f$$

$$cU\ \partial\mathbf{v}_s/\partial x = \mathbf{i}/F - (\mathbf{v}_f-\mathbf{v}_s)/F \tag{2}$$

$$\mathrm{grad}\ p = -(\mathbf{v}_f-\mathbf{v}_s)/F$$

where $D(\epsilon)$ has been replaced by $(1-\epsilon)/F$, since in the uniformly fluidized state $\mathbf{v}_s = 0$, $\mathbf{v}_f = \mathbf{i}$ and the second of (1) implies that $D(\epsilon) = (1-\epsilon)/F$. Note that only three of Equations (2) are independent; namely, the last two and one of the first two. The velocities, all lengths, and the pressure have been nondimensionalized, respectively, by using v_0, the minimum fluidization velocity, l, some typical length in the problem which will be specified later, and $\rho_s v_0^2 (1-\epsilon)$, where ρ_s is the density of the particles. The nondimensionless constants F and the bubble velocity U are

$$F = v_0^2/gl \qquad U = U_B/v_0 \tag{3}$$

From (2), $\mathbf{v}_f$ and $\mathbf{v}_s$ are irrotational and may be represented by the gradients of potential functions. We consider here the two-dimensional situation and introduce the potentials ϕ_f, ϕ_s, by

$$\mathbf{v}_f = \mathrm{grad}\ \phi_f \qquad \mathbf{v}_s = \mathrm{grad}\ \phi_s \tag{4}$$

Substituting (4) into the last two equations of (2) and integrating, we obtain

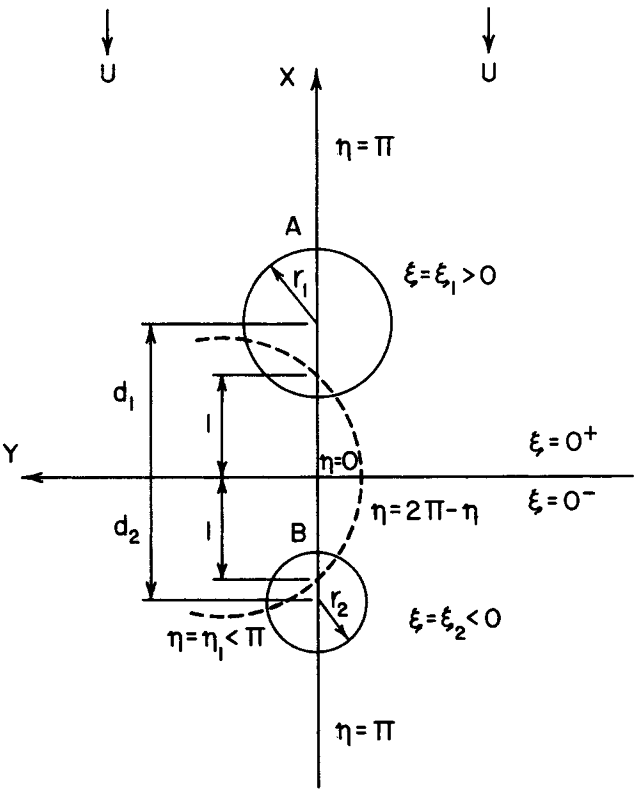

Fig. 1. Bipolar coordinates with circular cylinders of radii r_i = cosech ξ_i (i = 1, 2) with centers at x_i = coth ξ_i (i = 1, 2).

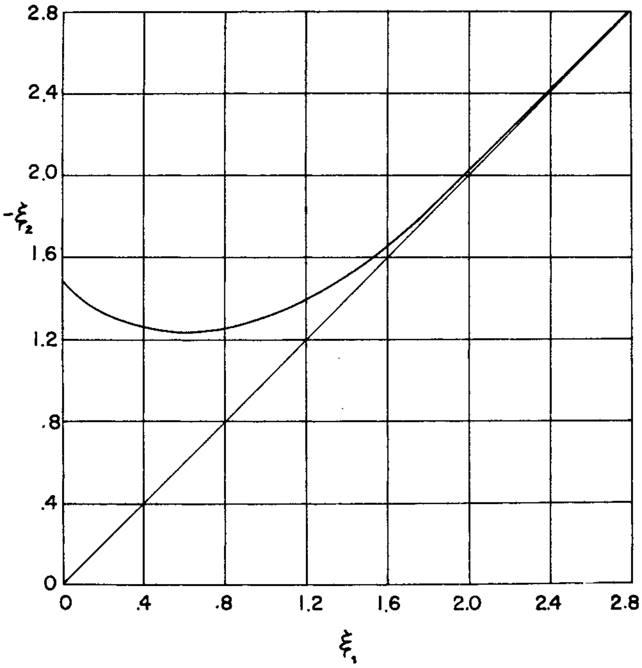

Fig. 2. Bubble dimensions (in bipolar coordinates) when the two bubbles move with the same velocity.

$$\phi_f = x + \phi_s - c\ F\ U^2\left[1+\frac{1}{U}\frac{\partial\phi_s}{\partial x}\right] \tag{5}$$

$$p - p_0 = -[\phi_f-\phi_s]/F \tag{6}$$

where, with the origin of coordinates at the center of the bubble or between them if there are two, we have used the boundary condition at infinity as

$$\phi_f \to x(1-U), \quad \phi_s \to -Ux, \quad p - p_0 \to x/F, \qquad \text{as}\quad x\to\infty \tag{7}$$

When ϕ_s is prescribed, as it must be in this theory, ϕ_f and hence p can be found immediately from (5) and (6).

For the situation represented by Figure 1, that of two bubbles moving up through a bed with the same velocity U, we require an expression for $\mathbf{v}_s$ or ϕ_s. We introduce bipolar coordinates [see for example, Morse and Feshbach (6)] ξ, η by

$$\zeta = \xi + i\eta = \log\frac{1+z}{1-z} \tag{8}$$

where $z = x + iy$ and the $\xi - \eta$ system shown in Figure 1 is dimensionless, since the distance between two poles is taken to be 2. It should be pointed out, however, that the use of the distance between poles as a characteristic length by no means fixes the distance between the bubbles. With vertical bubble motion, we consider the bubbles to be given by $\xi = \xi_1(>0)$ and $\xi = \xi_2(<0)$. Their radii r_1, r_2 are

$$r_i = \mathrm{cosech}\ \xi_1 \qquad i = 1, 2 \tag{9}$$

their centers are at

$$x_i = d_i = \coth\ \xi_i \qquad i = 1, 2 \tag{10}$$

and so the distance d between their centers is

$$d = d_1 + d_2 = (\coth\ \xi_1 - \coth\ \xi_2)^* \tag{11}$$

The particulate flow situation relative to the two bubbles is a uniform flow U in the negative x direction past the fixed bubbles and is given [see, for example, Morse and

* Thus, given r_1 and r_2, ξ_1, ξ_2, and d are determined.

Feshbach (6)] by the potential function $\phi_s(\xi, \eta)$ in bipolar coordinates as

$$\phi_s = -U \left[\begin{array}{ll} 1 + 2 \sum_1^\infty (-1)^n e^{-n\xi} \cos n\eta & \xi > 0 \\ -1 - 2 \sum_1^\infty (-1)^n e^{-n\xi} \cos n\eta & \xi < 0 \end{array} \right]$$
$$+ 2U \sum_1^\infty (-1)^n \cos n\eta \, [e^{n\xi} (e^{-2n\xi_2} - 1)$$
$$+ e^{-n\xi - 2n\xi_2} (e^{2n\xi_1} - 1)] \, [e^{2n\xi_1} - e^{2n\xi_2}]^{-1} \quad (12)$$

It is easy to check that (12) is the correct solution satisfying $\partial\phi_s/\partial\xi = 0$ on $\xi = \xi_i$, $i = 1, 2$ and such that $\mathrm{v}_s \rightarrow -U\,\mathrm{i}$ [see (7)] as $|x^2 + y^2| \rightarrow \infty$. Substitution of ϕ_s from (12) into (5) and then into (6) gives, respectively, the fluid-phase flow and the fluid-pressure distribution. The Davies and Taylor (3) method for obtaining U was used successfully by Jackson (5) and Murray (9) for fluidized systems. In the case treated here, we apply it two dimensionally, and it consists simply of expanding p on $\xi = \xi_1, \xi_2$ in powers of the angle subtended at the bubble centers measured from the leading edge of each bubble, that is A and B in Figure 1, and of requiring p to be constant to as high a power of the angle as possible. In this case, we simply substitute ϕ_s, from (12), in (5) and (6), and for the bubble with radius r_1, we expand p, on $\xi = \xi_1$, in a Taylor series about the point A, which here is $\eta = \pi$. In this way we can make p a constant to $0\,[(\eta - \pi)^4]$ by choosing cU to make the $0\,[(\eta - \pi)^2$ term zero. After a little algebra, this gives

$$\frac{1}{4cU^2F} = -2\,(\cosh\,\xi_1 + 1)^2 \sum_1^\infty n^2 \left[\frac{e^{n\xi_1} (e^{2n\xi_2} - 1)}{e^{2n\xi_1} - e^{2n\xi_2}} \right] \quad (13)$$

If we now expand p about the point B, which is $\eta = 0$, $\xi = \xi_2$, we choose cU so that the $0(\eta^2)$ term is zero, and in an exactly similar way to that which resulted in (13), we find

$$\frac{1}{4cU^2F} = -2(\cosh\,\xi_2 + 1)^2 \sum_1^\infty (-1)^n n^2 \left[\frac{e^{n\xi_2} (e^{2n\xi_1} - 1)}{e^{2n\xi_1} - e^{2n\xi_2}} \right] \quad (14)$$

If we consider the limiting cases when $r_1 \rightarrow 0$ with r_2 finite and $r_2 \rightarrow 0$ with r_1 finite, we get, as we must, the result given by Murray (9) for a single bubble. For example, in (13) let $r_2 \rightarrow 0$, with r_1 finite, which from (9) implies that $\xi_2 \rightarrow -\infty$ with ξ_1 finite, and we get

$$\frac{1}{4c_0U_0^2F} = 2(\cosh\,\xi_1 - 1)^2 \sum_1^\infty n^2\, e^{-n\xi_1} = \sinh\,\xi_1$$

where the subscript 0 denotes the value for a single bubble in isolation. With r_1 from (9), this gives

$$\frac{r_1}{4c_0U_0^2F} = 1 \quad (15)$$

In dimensional form (15) gives

$$U_{B_0} = \frac{1}{2}\,(gr'_1/c_0)^{1/2} \quad (16)$$

where the prime on r'_1 denotes the dimensional radius, and (16) is the result found by Murray (9).

Using (13) and (14), we can now find ξ_1, ξ_2, and the bubble radii and distance apart of their centers, when the bubbles rise up through a fluidized bed with the same dimensionless velocity U, by equating the right-hand sides of (13) and (14); namely

$$(\cosh\,\xi_1 - 1)^2 \sum_1^\infty n^2 \left[\frac{e^{n\xi_1} (e^{2n\xi_2} - 1)}{e^{2n\xi_1} - e^{2n\xi_2}} \right] = (\cosh\,\xi_2 + 1)^2 \sum_1^\infty (-1)^n n^2 \left[\frac{e^{n\xi_2} (e^{2n\xi_1} - 1)}{e^{2n\xi_1} - e^{2n\xi_2}} \right] \quad (17)$$

Analytically, from (17) it can be shown that one solution is $\xi_1 = \infty$, $\xi_2 = -\infty$, and another is $\xi_1 = 0$, ξ_2 a finite value. In general, however, (17) has to be solved numerically. This was done by assigning a $\xi_2(< 0)$ and then by using Newton's method to find ξ_1. The results are shown in Figure 2. With these values of ξ_1 and ξ_2, one can determine from (9) and (11) the corresponding radii of the bubble and the distances apart of their centers. The curve of r_2/r_1 against d/r_1 thus obtained is plotted in Figure 3. A numerical computation using (14) was carried out which showed that a slight increase in r_2 from the value which gives the same velocity for both bubbles leads, as expected, to an increase in the velocity of the follower. Therefore, it is plausible at this stage to state that for any bubble configuration which corresponds to a point in the region above the curve in Figure 3, a steady motion cannot be maintained, and that the following bubble will accelerate towards the leading bubble. Note that over a wide range of bubble radii this happens when $r_2 < r_1$. This is in agree-

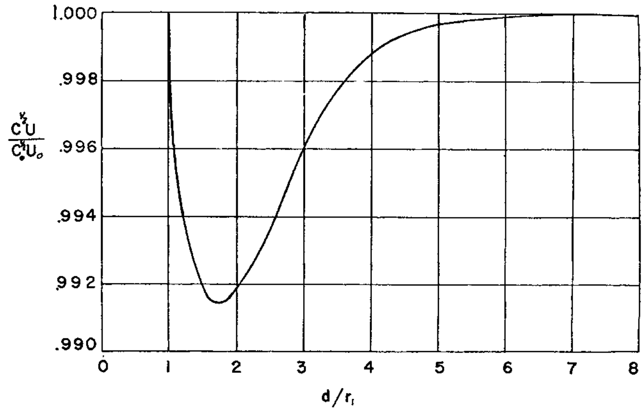

Fig. 4. Comparative velocities for the two-bubble (moving with the same velocity) case and that of a single bubble of radius r_1 as a function of the distance apart of the two bubbles.

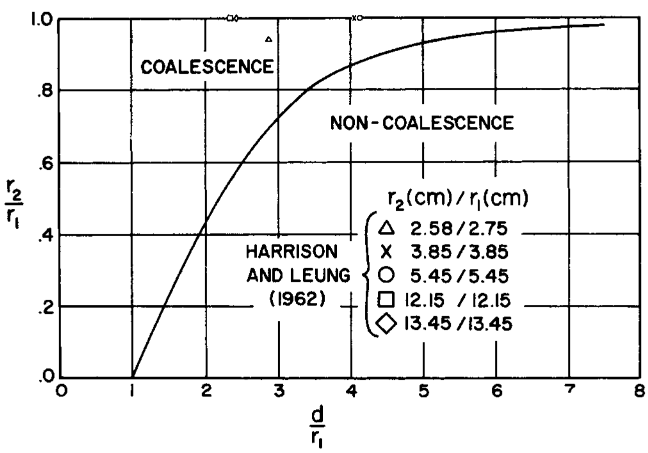

Fig. 3. Neutral coalescence curve: relation between the bubble radii and distance apart of their centers when they move with the same velocity. Toei and Matsuno's points lie on the $r_1 = r_2$ line.

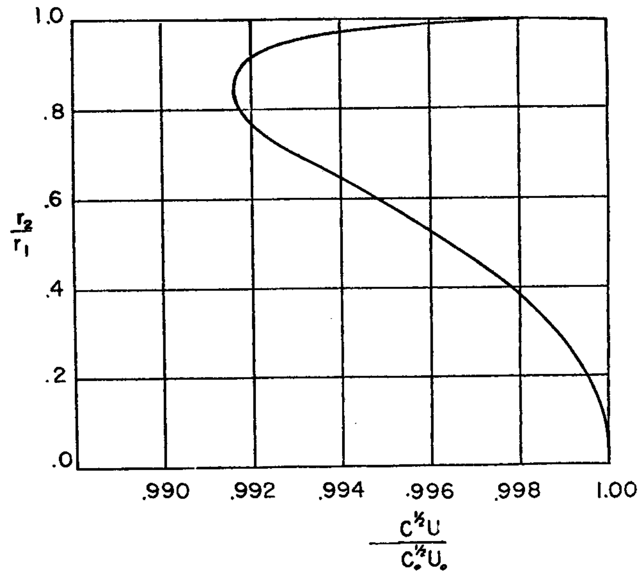

Fig. 5. Comparative velocities for the two-bubble case and that of a single bubble of radius r_1 as a function of the radii of the two bubbles.

ment with the results observed by Rowe and Partridge (*12*) who found that there was coalescence even when the lower bubble was of a smaller radius than the higher. If the point designated by given r_2/r_1 and d/r_1 lies below the curve, the bubbles in question will be unlikely to coalesce. The points in Figure 3 are obtained from the results of Harrison and Leung's (*4*) experiments on the coalescence of two two-dimensional bubbles moving in train up through a fluidized bed. Each point represents an experiment for which coalescence was observed. Note that all points lie in the region of coalescence in Figure 3. It is seen that the present theoretical results are not inconsistent with their experimental results.

Of more relevance here are the experimental results of Toei and Matsuno (*14*). They considered equal bubbles [which always coalesced in this situation (in train)] giving $r_2 = r_1$, which, in Figure 3 for example, is the top horizontal (parallel to d/r_1 axis) line which lies completely in the coalescing region. From their time-for-coalescence curve (their Figure 4), it seems, together with Figure 3 here, that the farther the experimental point lies from the coalescence curve, the faster will be the time for coalescence.

When the bubble radii are such that the two bubbles move with the same velocity, their velocity can be compared with that of one bubble in isolation. We shall take the higher bubble (with radius r_1) to use in the comparison with an isolated bubble of radius r_1. We thus divide (13) by sinh ξ_1, and from (9) this gives r_1 in the place of 1 on the left side. If we now compare this expression for cU^2, that is (13) divided by sinh ξ_1, with that from (15) with the same r_1 and F, we get, since v_0 is the same in both situations

$$\frac{c_0 U_{B0}^2}{c U_B^2} = \frac{c_0 U_0^2}{c U^2} =$$

$$- \frac{2(\cosh \xi_1 - 1)^2}{\sinh \xi_1} \sum_1^\infty n^2 \left[\frac{e^{n\xi_1}(e^{2n\xi_2} - 1)}{e^{2n\xi_1} - e^{2n\xi_2}} \right] \quad (18)$$

where U_{B0} and U_B are the dimensional velocities of the bubbles. The expression in (18) was evaluated numerically for given ξ_1, which from Figure 2 also assigns the corresponding ξ_2. These values also determine d (and r_1 and r_2). The results are given in Figures 4 and 5 which plot $c^{1/2}U/c_0^{1/2}U_0$ against d/r_1 and r_2/r_1 against $c^{1/2}U/c_0^{1/2}U_0$, respectively. Note that $c^{1/2}U/c_0^{1/2}U_0$ differs only slightly from unity which suggests, in this theory, that if the c's are the same, the velocity of two bubbles in train is only slightly smaller than that for the first bubble in isolation. Since Figures 3, 4, and 5 have coordinate axes as ratios, the various radii, bubble velocities, and distances may, of course, be given in any consistent dimensional terms.

In conclusion, the method suggested by Murray (9) provides some simply interpreted results for the motion of two bubbles moving in train up through a fluidized bed. For any given bubble radii and distance apart, Figure 3 suggests whether or not the two bubbles will coalesce.

ACKNOWLEDGMENT

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NOTATION

c = constant of optimization
d = dimensionless distance between the centers of two bubbles in train
$D(\epsilon)$ = drag coefficient
F = dimensionless number, Equation (3)
g = gravitational acceleration
$\mathbf{i}$ = unit vector in vertical x direction
p = fluid pressure
p_0 = constant
r'_1 = dimensional bubble radius
r_1 = dimensionless radius of the leading bubble
r_2 = dimensionless radius of the following bubble
U_B = dimensional bubble velocity
U = dimensionless bubble velocity
$\mathbf{v}_f$ = fluid velocity vector
$\mathbf{v}_s$ = solid-phase velocity vector

Greek Letters

ϵ = voidage
η = bipolar coordinate
ξ = bipolar coordinate
ρ_s = solid-phase density
ρ_f = fluid-phase density
ϕ_s = solid-phase velocity potential
ϕ_f = fluid-phase velocity potential

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